From asymptotic properties of general point processes to the ranking of financial agents

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Motivation

Market participants contribution to market quality

- To assess market quality, several metrics are used such as market depth, spread, aggressiveness and volatility.
- Disentangling market participants contribution to each of these metrics is possible, except for volatility.

How to build a model for the interactions between strategies of individual market participants and use it to assess their individual contribution to market volatility?

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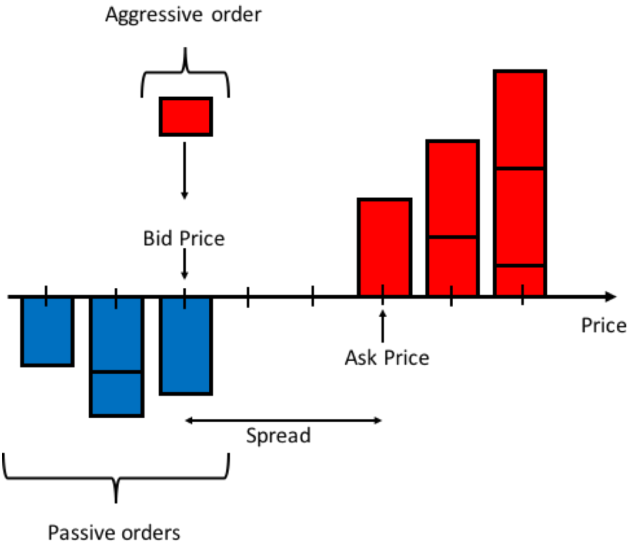
The Limit Order Book



Introduction to the model

Three elementary decisions

In our model, we only consider the price levels between the best bid and ask prices, and we assume that the agents can take three elementary decisions:

- Insert a limit order of a specific size, in average event size (AES[a]), at the best bid or ask price.
- Insert a buying or selling limit order of a specific size within the spread.
- Send a liquidity consuming order of a specific size at the best bid or ask price. Cancellation and market orders have the same effect on liquidity. Thus, they are aggregated to constitute the liquidity consumption orders.

[a] AES is the average size of events observed in the limit order book.

Order book dynamic

Notations

We use an event by event description. Each event is characterised by $(T_n, X_n) \in (\mathbb{R}^+, E)$ where:

- T_n is the time of the n^{th} event.
- X_n is a variable encoding the characteristics of the event:
  - **size** s_n**:** an integer representing the order size.
  - **price** p_n**:** equals to $k \in \mathbb{N}$ when the order is inserted at the price best bid $+k\tau_0$, where τ_0 is the tick size.
  - **direction** d_n**:** $+$ if it provides liquidity and $-$ when liquidity is removed.
  - **type** t_n^o**:** 1 (resp. 2) when the bid (resp. ask) is modified.
  - **agent** a_n**:** the market consists in N agents.

Order book dynamic

The order book state is modelled by the process $U_t = \left(Q_t^1, Q_t^2, S_t\right)$ where Q_t^1 (resp. Q_t^2) is the best bid (resp. ask) quantity and S_t is the spread.

Generalised intensity

General definition

The intensity $\lambda_t(e)$ is the instantaneous probability that an event of type $e \in E$ happens at t conditioned on the history of the market

$$\lambda_t(e) = \lim_{\delta t \to 0} \frac{\mathbb{P}[\#\{T_n \in (t, t+\delta t], X_n = e\} \geq 1 | \mathcal{F}_t]}{\delta t}.$$

The considered intensity in our model

$$\lambda_t(e) = \psi\Big(e, U_{t^-}, t + \sum_{0<T_i<t} \phi(e, U_{t^-}, t - T_i, X_i)\Big),$$

where

- ψ is a possibly non-linear function and is $\mathbb{R}_+$-valued function.
- ϕ is the Hawkes-like kernel representing the influence of the past events are is $\mathbb{R}_+$-valued function.
- U_t^- is the order book state relative to the last event before t.

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Market reconstitution

Market intensity

The market intensity $\lambda_t^M(e')$ of an event e' (e' does not contain the agent identity) in the exchange is given by

$$\lambda_t^M(e') = \sum_{a \leq N} \lambda_t\big((e', a)\big).$$

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Ergodicity

Theorem

Under suitable assumptions, $\bar{U}_t = (Q_t^1, Q_t^2, S_t, \lambda_t)$ is ergodic: there exists a probability measure μ such that

$$\lim_{t \to \infty} P_t(u, A) = P_0(\mu, A) \quad \forall u, A,$$

where u is an initial condition which is here a càdlàg function from $(-\infty, 0]$ into $(\mathbb{R}^+)^4$.

In this ergodic setting, we can derive asymptotic results for long-term behaviour of our system.

Scaling limit

The reference price after n jumps P_n satisfies $P_n = P_0 + \sum_{i=1}^{n} \Delta P_i$ where $\Delta P_i = P_i - P_{i-1} = \eta_i$ and $\mathbb{E}[\eta_i] = 0$. We assume $\eta_i = f(U_i)$ and denote by

$$X_n(t) = \frac{P_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad \forall t \geq 0.$$

Theorem

Under the stationary distribution, the quantity $X_n(t)$ satisfies the following convergence result:

$$X_n(t) \xrightarrow{\mathcal{L}} \sigma W_t,$$

with $\sigma^2 = \mathbb{E}_\mu[\eta_0^2] + 2\sum_{k\geq 1} \mathbb{E}_\mu[\eta_0 \eta_k]$ and μ the stationary distribution.

Stationary probability computation

Theorem

The stationary distribution of the process U_t, denoted by π, satisfies

$$\pi Q = 0$$
$$\pi \mathbf{1} = 1.$$

where the infinite dimensional matrix Q verifies

$$Q(z, z') = \sum_{e \in E(z,z')} \mathbb{E}_\mu[\lambda(e)\|],$$

with $E(z, z')$ the set of events directly leading to z' from z.
The matrix Q can be estimated the following way:

$$\hat{Q}(u, u') = \frac{N_t^{u,u'}}{t^u} \underset{t \to \infty}{\to} Q(u, u'), \quad a.s.$$

Note that the form of the estimator $\hat{Q}(u, u')$, and hence π, does not depend on the model.

The Markov case

Volatility computation in the Markov case

In the Markov case, the quantity to compute becomes

$$\sigma^2 = \mathbb{E}_\pi[\eta_0^2] + 2\sum_{k\geq 1}\mathbb{E}_\pi[\eta_0\eta_k].$$

We define P the Markov chain associated to U such that

$$P_{u,u'} = \begin{cases} -Q_{u,u'}/Q_{u,u} & \text{if } u \neq u' \text{ and } Q_{u,u} \neq 0 \\ 0 & \text{if } u \neq u' \text{ and } Q_{u,u} = 0, \end{cases}$$

$$P_{u,u} = \begin{cases} 0 & \text{if } Q_{u,u} \neq 0 \\ 1 & \text{if } Q_{u,u} = 0, \end{cases}$$

with $P_{u,u'}$ the transition probability from u to u' after one jump.

$$\mathbb{E}_\pi[\eta_0\eta_k] = \sum_u \pi(u)f(u)\mathbb{E}_u[\eta_k], \quad \mathbb{E}_u[\eta_k] = \sum_{u'} P^k_{u,u'}f(u').$$

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Framework

Data description

- Data provided by the french regulator AMF.
- We study four large tick European stocks: Air Liquid, EssilorLuxottica, Michelin and Orange, on Euronext, over a year period: from January 2017 till December 2017.
- Model simplifications: all the orders have same size, spread equal to one tick and simple queue-reactive case.
- In this setting, the events are only, for both limits, to increase by one unit or decrease by one unit.
- The matrix Q is therefore a function of the arrival intensity of limit orders on the one hand; market orders and cancellations on the other hand.

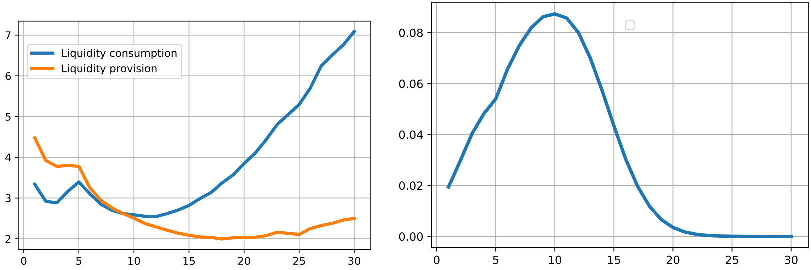
Preliminary statistics

Asset	Number of insertion orders (in millions of orders)	Number of cancellation orders (in millions of orders)	Number of aggressive orders (in millions of orders)	Ratio of cancellation orders number over aggressive orders number	Average spread (in ticks)
Air Liquide	2.36	2.40	0.21	11.4	1.07
EssilorLuxottica	3.90	3.96	0.34	11.6	1.11
Michelin	3.81	4.01	0.32	12.5	1.14
Orange	6.60	6.66	0.47	14.1	1.14

Table: Preliminary statistics on the assets.

Results relative to Air Liquide

(a) Intensity of the market | (b) Stationary measure Q^1



Long-term price volatility $\sigma^2 = 0.227$.

Figure: (a) Liquidity insertion and consumption intensities (in orders per second) with respect to the queue size (in average event size) and (b) the corresponding stationary distribution of (Q^1) with respect to the queue size (in average event size), proper to Air Liquide.

The expected volatility in case of withdrawal of a market maker

Intensities and $\sigma_{10}^{2,M}$ when one market maker leaves the market

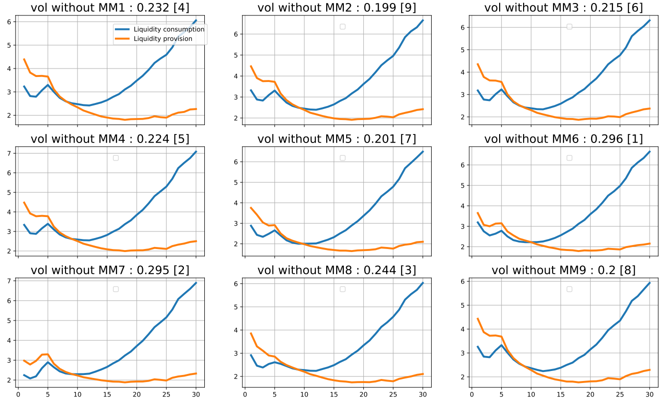


Figure: Liquidity insertion and consumption intensities (in orders per second) with respect to the queue size (in AES) and $\sigma_{10}^{2,M}$ when one market maker is ejected from the market for the stock Air Liquide.

Ranking of the market makers

Market maker	Ranking Air Liquide	Market share Air Liquide	Ranking ExilorLux-ottica	Market share ExilorLux-ottica	Ranking Michelin	Market share Michelin	Ranking Orange	Market share Orange
MM1***	4	4%	3	3%	3	4%	3	3%
MM2	9	1%	9	1%	9	1%	7	1%
MM3	6	5%	6	5%	7	4%	5	4%
MM4	5	1%	4	1%	4	0%	4	1%
MM5	7	5%	8	5%	8	5%	9	5%
MM6****	1	3%	2	3%	1	3%	1	4%
MM7****	2	7%	1	12%	2	9%	2	7%
MM8*	3	9%	5	5%	5	5%	6	4%
MM9	8	2%	7	2%	6	2%	8	2%

Table: Market share and ranking of markets makers.